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ON STABILITY OF NONLINEAR OSCILLATIONS OF A MEMBRANE

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The Liapunov method is used in the function space to study the vertical oscillations of a membrane, for nonlinear law of elasticity and finite deformations.

1. Formulation of the problem. We assume that the membrane in the undeformed state occupies the region Ω in the Ox_1x_2-plane, and is in its natural state. The deformation of the membrane consists of uniform stretching along the Ox_1 and Ox_2 axes and deflection u along the Ox_3-axis, with its boundary Γ remaining in the Ox_1x_2-plane, i. e.

$$x_1 = (1+e)a_1, \quad x_2 = (1+e)a_2, \quad x_3 = u(a_1, a_2, t) \quad (e > 0) \tag{1.1}$$
$$u(a_1, a_2, t)\,|_\Gamma = 0, \quad \Gamma = \partial\Omega$$

We assume that the potential energy of deformation depends only on the principal extensions of the dilatation tensor. The kinetic energy of the membrane is given, in accordance with the first assumption, by

$$T = \frac{1}{2}\rho\int_\Omega u^{\cdot 2}da \quad \left(u^{\cdot} = \frac{\partial u}{\partial t}, \ da = da_1 da_2\right)$$

where ρ is the surface tension of the membrane, and is assumed constant. From (1.1) it follows that the principal extensions of the dilatation tensor have the form

$$\lambda_1 = 1+e, \quad \lambda_2 = [(1+e)^2 + u_1^2 + u_2^2]^{1/2} \quad (u_1 = \partial u / \partial a_1, \ u_2 = \partial u / \partial a_2)$$

According to the last assumption, the potential energy of the deformations can be written in the form of a functional

$$\mathrm{E}[u] = \int_\Omega F(\xi)\,da, \quad \xi = \lambda_2 \tag{1.2}$$

Here $F(\xi)$ is a function depending on the properties of the membrane material. In particular, when $F(\xi) = \xi^2 - (1+e)^2$, the problem of membrane oscillations becomes linear (see e.g. [1]). The function

$$F(\xi) = b_2\xi^2 + b_1\xi + b_0 \quad (b_1, b_2, b_0 = \mathrm{const}) \tag{1.3}$$

corresponds to the Hooke's Law and the function

$$F(\xi) = \alpha\left[\xi^2 + \frac{1}{(1+e)^2\xi^2} + (1+e)^2 - 3\right] + \beta\left[(1+e)^2\xi^2 + \frac{1}{\xi^2} + \frac{1}{(1+e)^2} - 3\right] \tag{1.4}$$

where α and β are constants, describes a model of incompressible rubber proposed by R. Rivlin and D. Saunders [2].

The configurational space of the system V is represented by a linear space of functions defined on Ω and vanishing at the boundary

$$V = \{u:\ \mathrm{E}\,[u] < \infty,\ u\,|_\Gamma = 0\}$$

The properties of the space depend on the form of the function $F(\xi)$. Thus in (1.3) and (1.4) $V = \overset{\circ}{W}{}_2^1(\Omega)$. In particular, we have

$$V = \overset{\circ}{W}{}_4^1(\Omega) \text{ when } F(\xi) = \sum_{k=0}^{4} b_k \xi^k$$

The phase space of the system is defined as a straight product of the configurational space V and the Hilbert space of velocities H

$$H = \{u^\cdot : u^\cdot \in L_2(\Omega),\ u^\cdot\,|_\Gamma = 0\}$$

The variational equation of motion of the membrane (the d'Alembert — Lagrange principle) is written in the form

$$\rho\,(u^{\cdot\cdot},\ \delta u) + (\nabla \mathrm{E}\,[u],\ \delta u) - (f,\ \delta u) = 0, \quad \forall \delta u \in V \tag{1.5}$$

$$(u^{\cdot\cdot},\ \delta u) \equiv \int_\Omega u^{\cdot\cdot} \delta u\, da, \quad (\nabla \mathrm{E}\,[u],\ \delta u) \equiv \int_\Omega \frac{\partial F}{\partial \xi}\frac{1}{\xi}(u_1 \delta u_1 + u_2 \delta u_2)\, da$$

where $f(a_1, a_2)$ denote the external surface forces which we assume to be time independent.

2. Stationary solution and its stability. Equation (1.5) has a stationary solution $u = u_0 \in V$, which describes the position of equilibrium and satisfies the condition

$$(\nabla \mathrm{E}\,[u_0],\ \delta u) - (f,\ \delta u) = 0, \quad \forall \delta u \in V \tag{2.1}$$

Let us inspect the stability of the solution u_0. The first integral of (1.5) (Energy Conservation Law) has the form

$${}^1/_2 \rho\,(u^\cdot,\ u^\cdot) + \mathrm{E}\,[u] - (f,\ u) = E \tag{2.2}$$

Replacing in (2.2) u by $u + u_0$, we obtain

$${}^1/_2 \rho\,(u^\cdot,\ u^\cdot) + \mathrm{E}\,[u_0 + u] - (f,\ u_0) - (f,\ u) = E \tag{2.3}$$

where u is the deviation of the membrane from its position of equilibrium.

Let us consider, in the phase space of the system, the functional

$$\Phi\,[u^\cdot,\ u] = {}^1/_2 \rho\,(u^\cdot,\ u^\cdot) + \mathrm{E}\,[u_0 + u] - \mathrm{E}\,[u_0] - (f,\ u) \tag{2.4}$$

When $u = u^\cdot = 0$, the functional (2.4) vanishes. Let us assume that $\Phi\,[u^\cdot,\ u]$ is positive-definite and bounded in some neighborhood of the zero of the phase space (i.e. in terms of the initial notation, in the neighborhood of $(\dot{0}, u_0)$), and choose it as the Liapunov functional the derivative of which is equal to zero by virtue of the equations of motion. Thus the stationary solution (2.1) is Liapunov stable [3], the quantity ${}^1/_2\rho$ $(u^\cdot,\ u^\cdot)$ is positive-definite and bounded on the space of velocities ($(u^\cdot,\ u^\cdot)$ denotes the square of the norm).

We shall show now that the functional (2.4) is a Liapunov functional for the elasticity laws (1.3) and (1.4). We select the norm of the configurational space $\overset{\circ}{W}{}_2^1(\Omega)$ in the form

$$\|u\| = \left[\int_\Omega (u_1^2 + u_2^2)\, da\right]^{1/2}$$

L e m m a. If the second Fréchet variation of the functional $E[u]$ is positive-definite and bounded in some neighborhood of the stationary solution, i.e.

$$k_1 \| u \|^2 \leqslant (\nabla^2 E[v]u, u) \leqslant k_2 \| u \|^2 \quad (k_1 > 0) \tag{2.5}$$

for $\| v - u_0 \| < h \ (h > 0)$ where $\| \cdot \|$ denotes the norm in the configurational space V, then for $\| u \| < h$ we have

$$\tfrac{1}{2}k_1 \| u \|^2 \leqslant E[u + u_0] - (\nabla E[u_0], u) - E[u_0] \leqslant \tfrac{1}{2}k_2 \| u \|^2 \tag{2.6}$$

P r o o f. We introduce the function $W(\tau) : [0, 1] \to R^1$ defined by the relations

$$W(\tau) = E[u_0 + \tau u] - (\nabla E[u_0], \tau u) - E[u_0] \tag{2.7}$$
$$dW / d\tau = (\nabla E[u_0 + \tau u] - \nabla E[u_0], u)$$
$$d^2W / d\tau^2 = (\nabla^2 E[u_0 + \tau u]u, u)$$

If $\| u \| < h$, then according to (2.5)

$$k_1 \| u \|^2 \leqslant d^2W / d\tau^2 \leqslant k_2 \| u \|^2 \tag{2.8}$$

Integrating the inequality (2.8) twice with respect to τ and taking into account the relations $W(0) = dW(0) / d\tau = 0$, we arrive at (2.6).

In the case of (1.3), the second Fréchet variation of the functional $E[u]$ has the form

$$(\nabla^2 E[v]u, u) = \int_\Omega \left[\left(2b_2 - \frac{b_1}{\xi} \right) (u_1^2 + u_2^2) + \frac{b_1}{\xi^3} (v_1 u_1 + v_2 u_2)^2 \right] da,$$
$$\xi^2 = (1 + e)^2 + v_1^2 + v_2^2$$

Then the following estimate holds:

$$[2b_2 - b_1 / (1 + e)] \| u \|^2 \leqslant (\nabla^2 E[u_0]u, u) \leqslant (2b_2 + \tfrac{3}{2}b_1) \| u \|^2$$

We express the coefficients b_2 and b_1 in terms of the shear modulus G and of the transverse compression coefficient m (Poisson's ratio) [4] as follows:

$$b_2 = Gm / (m - 1), \quad b_1 = 2(m - e)G / (m - 1)$$

From this it follows that $2b_2 - b_1 / (1 + e) > 0$ and, according to the Lemma, the functional (1.3) is positive-definite and bounded and hence the stationary solution is stable.

We shall show that analogous inequalities also hold for the functional (1.4). Consider the function $W(\tau)$ appearing in (2.7) and constructed with help of the functional (1.4). We have

$$(\nabla E[v]u, u) = \int_\Omega 2 \left\{ \alpha \left[1 - \frac{1}{(1+e)^2 \xi^4} \right] + \beta \left[(1+e)^2 - \frac{1}{\xi^4} \right] \right\} (v_1 u_1 + v_2 u_2)\, da \tag{2.9}$$

$$(\nabla^2 E[v]u, u) = \int_\Omega \left\{ \left[\frac{\alpha}{(1+e)^2} + \beta \right] \frac{8}{\xi^6} (v_1 u_1 + v_2 u_2)^2 + \left[\alpha \left(1 - \frac{1}{(1+e)^2 \xi^4} \right) + \right. \right.$$
$$\left. \left. \beta \left((1+e)^2 - \frac{1}{\xi^4} \right) \right] 2 (u_1^2 + u_2^2) \right\} da, \quad \xi^2 = (1+e)^2 + v_1^2 + v_2^2$$

Since $\xi^4 \geqslant (1 + e)^4$, the estimate of the second relation of (2.9) is given by the inequality

$$(\nabla^2 E\,[v]\,u,\,u) \geqslant c_1 \|u\|^2, \quad c_1 = 2\alpha\left[1 - \frac{1}{(1+e)^6}\right] + 2\beta\left[(1+e)^2 - \frac{1}{(1+e)^4}\right] > 0 \qquad (2.10)$$

The boundedness of the second variation (2, 9) in $W^{\circ}{}_2{}^1\,(\Omega)$ follows from the estimate

$$(\nabla^2 E\,[v]\,u,\,u) \leqslant c_2 \|u\|^2, \quad c_2 = \frac{8\alpha}{(1+e)^6} + \frac{8\beta}{(1+e)^4} + 2\alpha + 2\beta\,(1+e)^2 \qquad (2.11)$$

From the inequalities (2, 10) and (2, 11) we conclude that the functional (2, 4) is positive-definite and bounded, therefore the stationary solution of (1, 5) is Liapunov stable.

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